Density Evolution and LDPC Convolutional Codes

Lentmaier *et al.* (2009) – Approaching Capacity with Asymptotically Regular LDPC Codes

Urbanke (2010) – Spatially Coupled Codes – A New Paradigm for Code Design (Talk at KTH)

Christian Häger

Department of Signals and Systems
Communication Systems Group
Chalmers University of Technology
Gothenburg, Sweden

ECC Seminar – May 29, 2012

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Outline

Recap: Density Evolution

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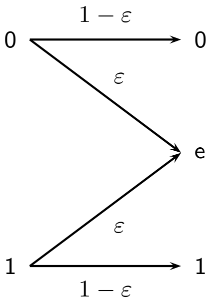
- Consider the binary erasure channel:



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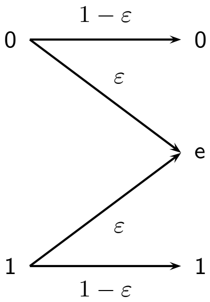

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- Here, density evolution is particularly simple: Track the average probability $p^{(l)}$ of an erasure after l iterations.
  - $(l = 0)$: $0\ 1\ 0\ e\ 1\ 0\ 1\ e\ e\ 1\ 0\ \ldots$ $\quad p^{(0)} = \varepsilon$
  - $(l = 1)$: $0\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ e\ 1\ 0\ \ldots$ $\quad p^{(1)} = ?$
  - $\ldots$

Recall the Sum-Product Algorithm

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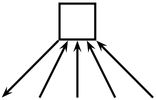

- Check node (CN) update:

$$L_{i\to j} = 2\tanh^{-1}\left(\prod_{j'\in N(i)-\{j\}} \tanh\left(\frac{1}{2}L_{j'\to i}\right)\right)$$

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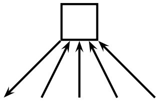

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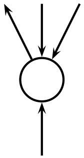

- Variable node (VN) update:

$$L_{j\to i} = L_j + \sum_{i'\in N(j)-\{i\}} L_{i'\to j}$$

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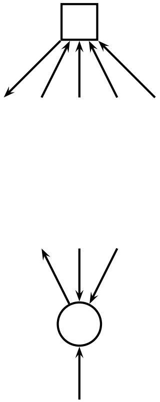
- $p^{(l)}$: probability that an outgoing VN message is an erasure, with $p^{(0)} = \varepsilon$



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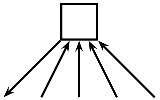

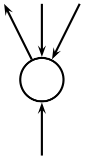

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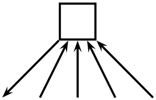
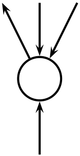
$$q^{(l)} = 1 - (1 - p^{(l-1)})^{k-1}$$





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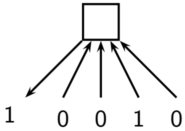

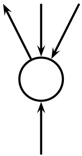

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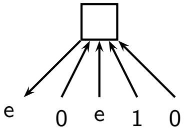

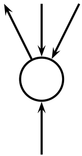

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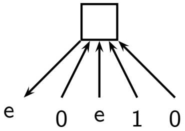

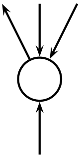

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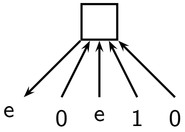

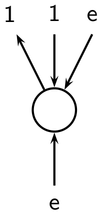

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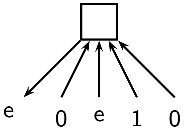

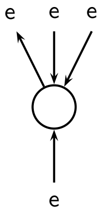

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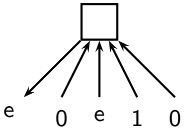

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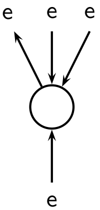


- Averaged over all VNs and CNs ($\lambda(x)$ is the VN degree distribution, $\rho(x)$ is the CN degree distribution):

$$p^{(l)} = \varepsilon \lambda(1 - \rho(1 - p^{(l-1)}))$$

Threshold Example (Capacity $C = 0.5$)

Example

- Regular $(3, 6)$ LDPC block code ensemble, rate $R = 1/2$:

$$\rho(x) = x^5 \qquad \lambda(x) = x^2$$

- Threshold $\varepsilon^* = 0.429$

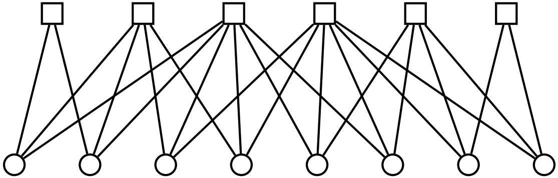
Example

- Regular $(5, 10)$ LDPC block code ensemble, rate $R = 1/2$:

$$\rho(x) = x^9 \qquad \lambda(x) = x^4$$

- Threshold $\varepsilon^* = 0.341$

Density Evolution for Protograph Ensembles



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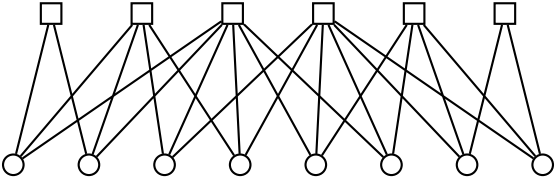

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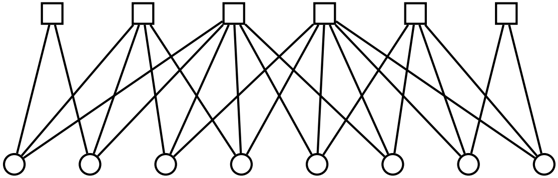

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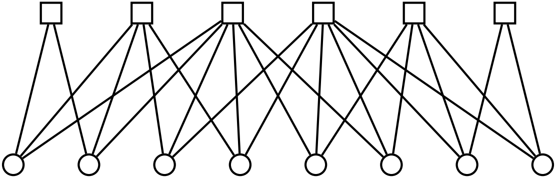

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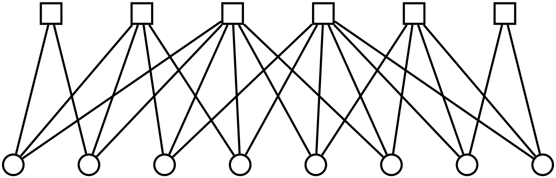
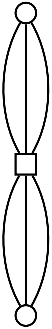


- Protograph = prototype graph
- Tanner graph of the code results from a copy-and-permute procedure that preserves the degree distribution of all nodes
- Imposes structure on the resulting LDPC code ensemble
- Now the erasure probabilities are functions of the edges:

$$q^{(l)}(e_{i\to j}) = 1 - \prod_{j'\in N(i)-\{j\}} (1 - p^{(l-1)}(e_{j'\to i}))$$

$$p^{(l)}(e_{j\to i}) = \varepsilon \prod_{i'\in N(i)-\{j\}} q^{(l)}(e_{i'\to j})$$

Spatial Coupling via "Edge Spreading"



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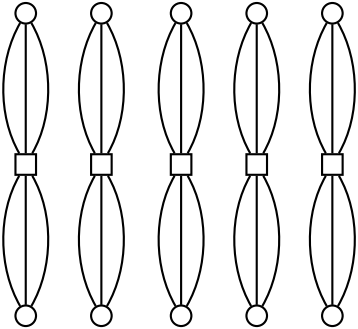

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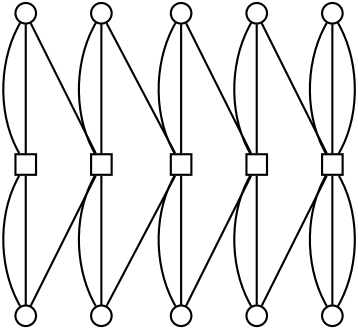

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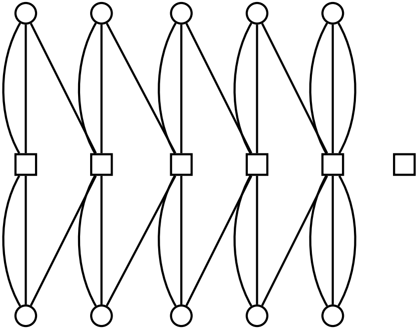

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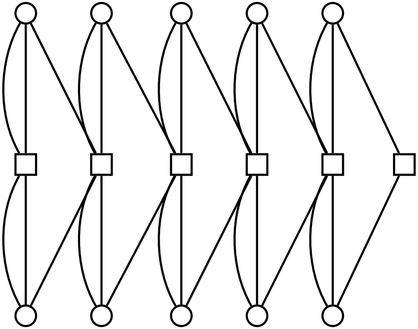

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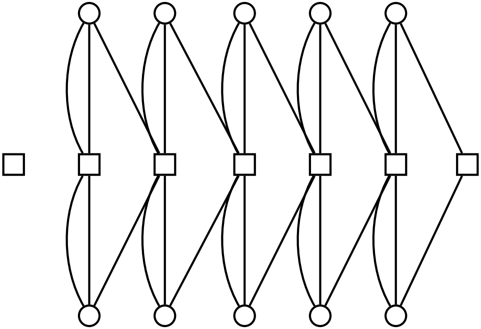

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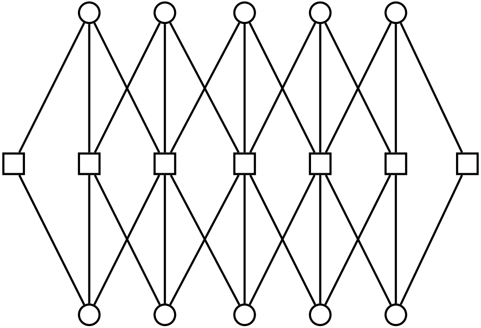

Properties

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- Parity check matrix has a convolutional-like structure:

$$B = \begin{pmatrix} 1&1&0&0&0&0&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0&0&0\\ 1&1&1&1&1&1&0&0&0&0&0&0&0&0\\ 0&0&1&1&1&1&1&1&0&0&0&0&0&0\\ 0&0&0&0&1&1&1&1&1&1&0&0&0&0\\ 0&0&0&0&0&0&1&1&1&1&1&1&0&0\\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1\\ 0&0&0&0&0&0&0&0&0&0&1&1&1&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1 \end{pmatrix}$$

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- Slight irregularities of the CN degrees at the beginning and end of the protograph
- However, as the number of protograph copies L grows large, the code becomes asymptotically regular and the rate loss becomes negligable

Thresholds for Protograph-based LDPC Convolutional Codes

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- Based on (3,6) LDPC Block protograph ($\varepsilon^* = 0.429$):

L	R_L	ε_L^*	$\varepsilon_{Sh}(R_L)$
5	0.300	0.587	0.700
10	0.400	0.504	0.600
20	0.450	0.488	0.550
40	0.475	0.488	0.525
∞	0.500	0.488	0.500

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- Based on (5,10) LDPC Block protograph ($\varepsilon^* = 0.341$):

L	R_L	ε_L^*	$\varepsilon_{Sh}(R_L)$
5	0.100	0.625	0.900
10	0.300	0.512	0.700
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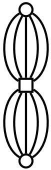

Erasure Probability Evolution at $\varepsilon = 0.483$, $L = 20$, (3,6) LDPC code

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- Spatial coupling and termination leads to slight irregularities of the CN degrees
- Same effect is also present for other channels, e.g., AWGN

Thank you!