Deterministic and Ensemble-Based Spatially-Coupled Product Codes

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In This Talk . . .

- Deterministic code construction that recovers product codes, staircase codes, and block-wise braided codes as special cases
- Rigorous density evolution analysis possible over the binary erasure channel
- Application: Spatially-coupled product codes

Introduction: Product Codes and Staircase Codes

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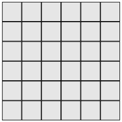
rectangular array [Elias, 1954]



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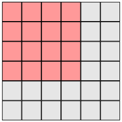

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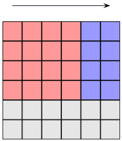

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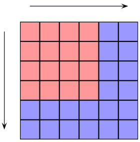

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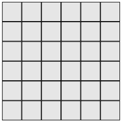

each row/column is a codeword in
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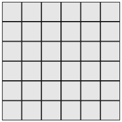

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Tanner
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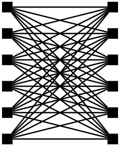

constraint node degree = component code length

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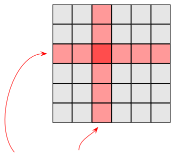

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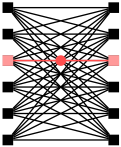

edge = degree-2 variable node

constraint node degree = component code length

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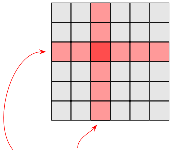
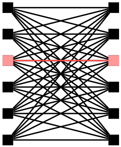


each row/column is a codeword in some component code

Tanner graph



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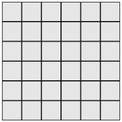

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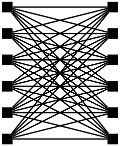

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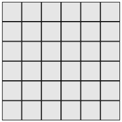

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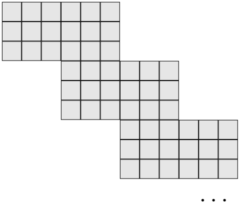

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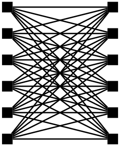

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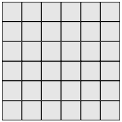

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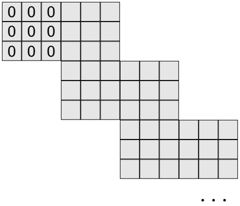

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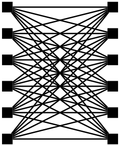

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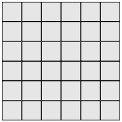

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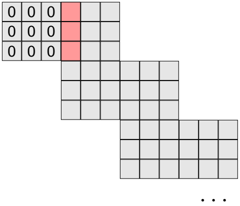

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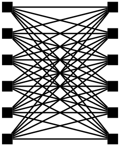

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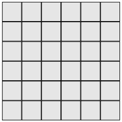

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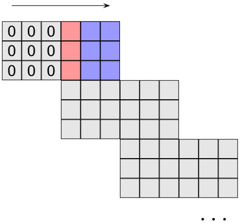

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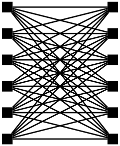

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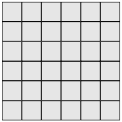

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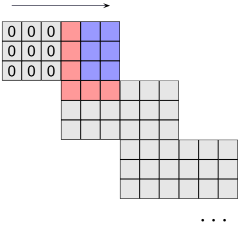

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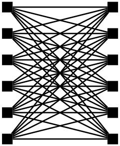

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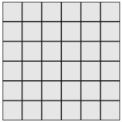

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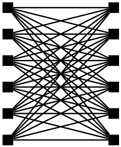

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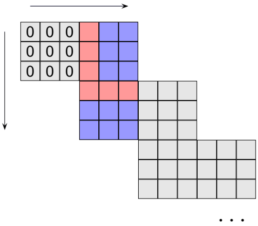

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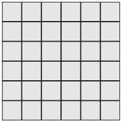

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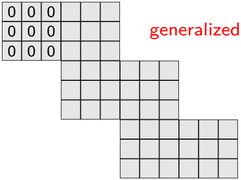

generalized product code (GPC)

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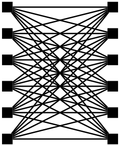

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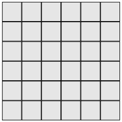

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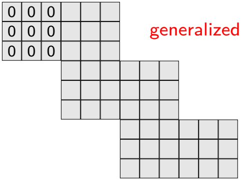

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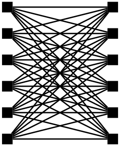

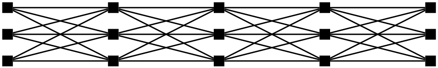

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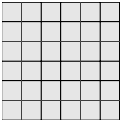

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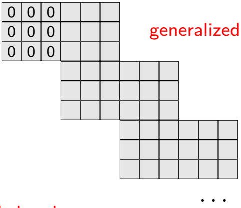

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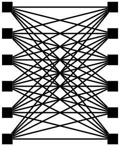
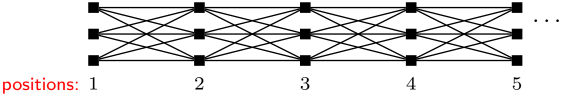


spatially-coupled code



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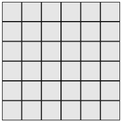

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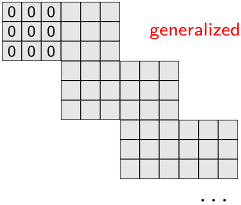

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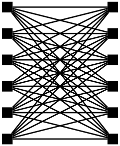

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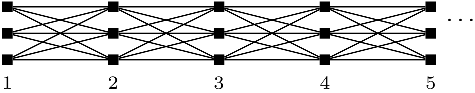
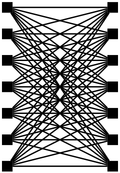
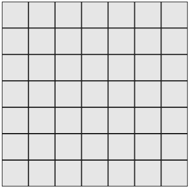


- Deterministic codes with fixed and structured Tanner graph

Iterative Bounded-Distance Decoding





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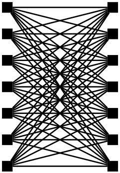

0	1	0	1	0	1	0
0	1	0	1	1	0	1
0	1	0	1	0	1	0
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	0	0	1	1	1
0	1	0	0	0	1	1

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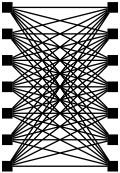
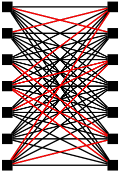


0	?	0	?	0	1	?
?	1	0	1	1	0	1
0	1	0	?	0	?	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	?	?	1	1	?
0	1	0	?	0	1	1

- Codeword transmission over binary erasure channel with erasure probability p

Iterative Bounded-Distance Decoding



0	?	0	?	0	1	?
?	1	0	1	1	0	1
0	1	0	?	0	?	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	?	?	1	1	?
0	1	0	?	0	1	1

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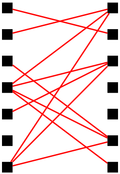

residual graph

0	?	0	?	0	1	?
?	1	0	1	1	0	1
0	1	0	?	0	?	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	?	?	1	1	?
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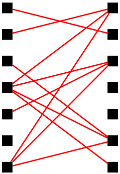

residual graph

0	?	0	?	0	1	?
?	1	0	1	1	0	1
0	1	0	?	0	?	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	?	?	1	1	?
0	1	0	?	0	1	1

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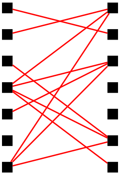
- Each constraint node corresponds to t-erasure correcting component code

Iterative Bounded-Distance Decoding



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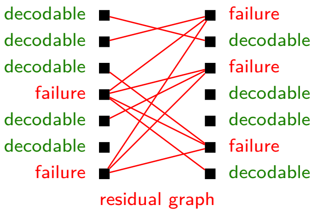
0	?	0	?	0	1	?
?	1	0	1	1	0	1
0	1	0	?	0	?	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	?	?	1	1	?
0	1	0	?	0	1	1

- Codeword transmission over binary erasure channel with erasure probability p
- Each constraint node corresponds to t-erasure correcting component code
- ℓ iterations of bounded-distance decoding $=$ peeling of vertices with degree $\leq t$ (in parallel)

Iterative Bounded-Distance Decoding

1st iteration $(t = 2)$



0	?	0	?	0	1	?
?	1	0	1	1	0	1
0	1	0	?	0	?	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	?	?	1	1	?
0	1	0	?	0	1	1

- Codeword transmission over binary erasure channel with erasure probability p
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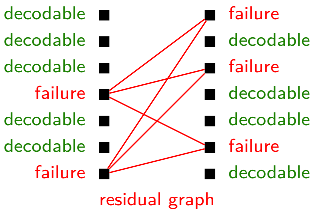

0	1	0	?	0	1	?
0	1	0	1	1	0	1
0	1	0	?	0	1	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
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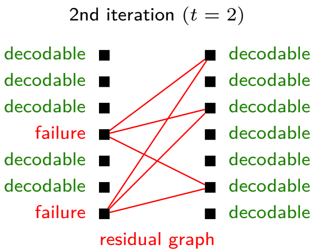

0	1	0	?	0	1	?
0	1	0	1	1	0	1
0	1	0	?	0	1	?
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	0	?	1	1	?
0	1	0	0	0	1	1

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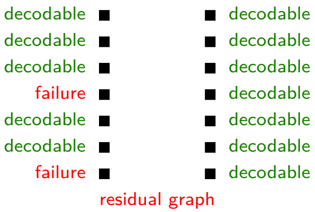
Iterative Bounded-Distance Decoding

2nd iteration $(t = 2)$



0	1	0	1	0	1	0
0	1	0	1	1	0	1
0	1	0	1	0	1	0
1	1	1	0	1	1	0
0	0	1	0	0	0	1
1	0	0	0	1	1	1
0	1	0	0	0	1	1

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Performance Prediction

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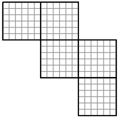

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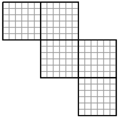

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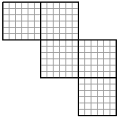

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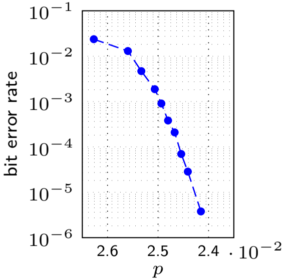

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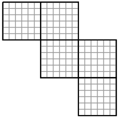

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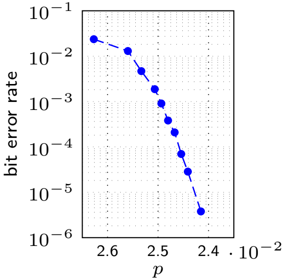

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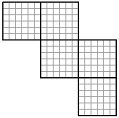

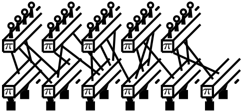

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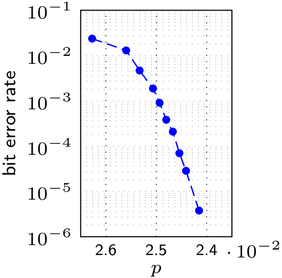

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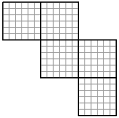

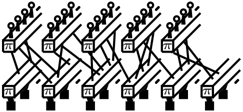

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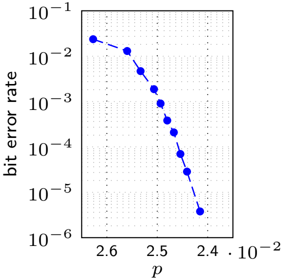

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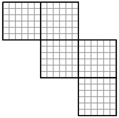

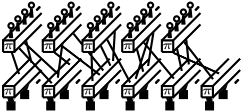

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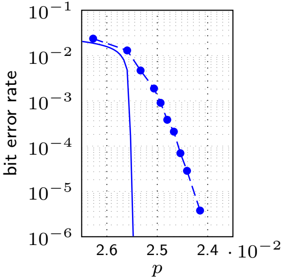

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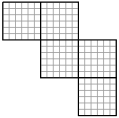

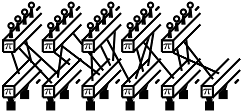

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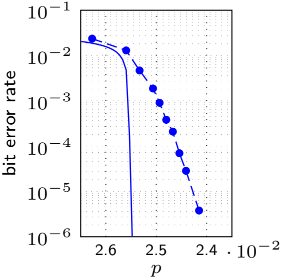
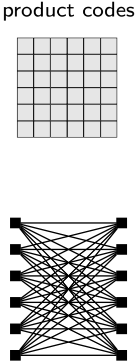
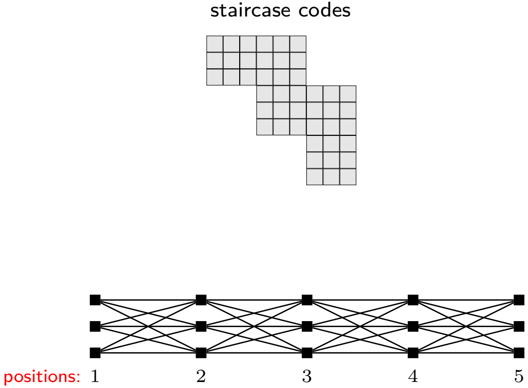


Fundamental question

Is it possible to directly analyze deterministic GPCs without the detour to code ensembles?

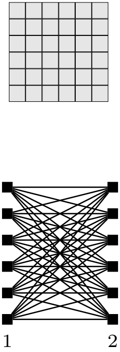
Parametrized Construction of Generalized Product Codes



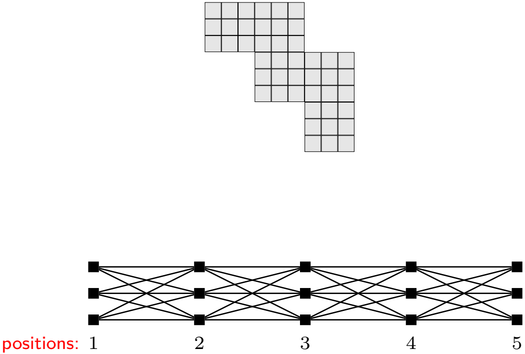


Parametrized Construction of Generalized Product Codes

product codes



staircase codes



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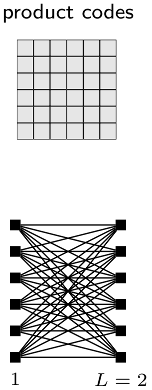

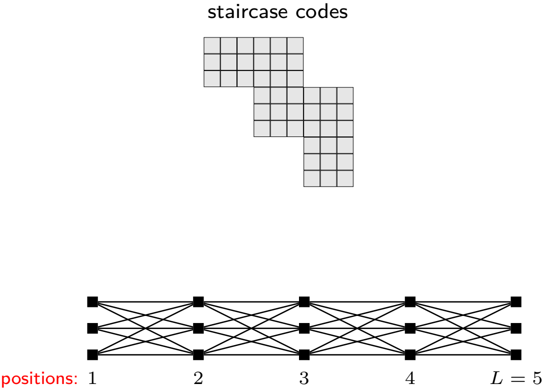

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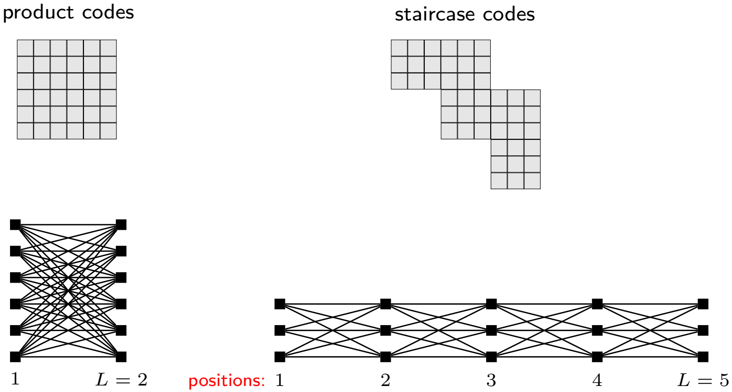
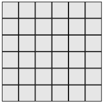


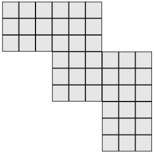
$\boldsymbol{\eta}$: symmetric $L \times L$ matrix that defines graph connectivity

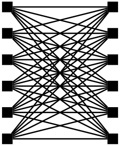
Parametrized Construction of Generalized Product Codes

product codes

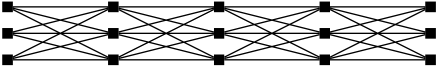


staircase codes





1 $L = 2$



positions: 1 2 3 4 $L = 5$

$$\boldsymbol{\eta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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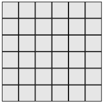
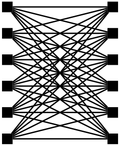
Parametrized Construction of Generalized Product Codes

product codes





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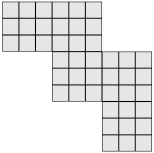
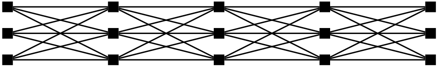
staircase codes





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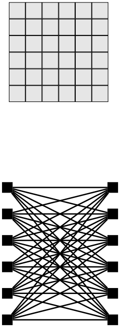
product codes



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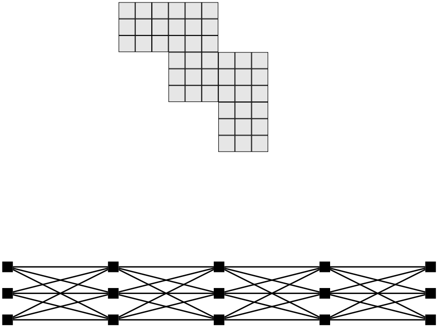
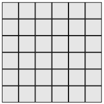


Parametrized Construction of Generalized Product Codes

product codes



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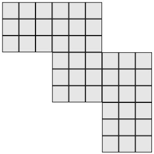
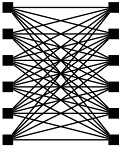
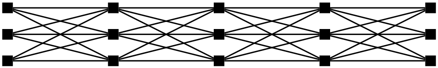


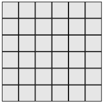


n: "problem size", proportional to the number of constraint nodes



Parametrized Construction of Generalized Product Codes

product codes



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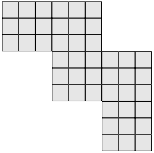

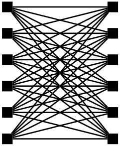

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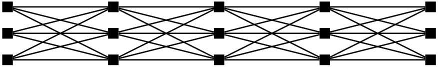
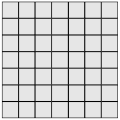


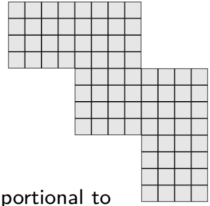
increasing n

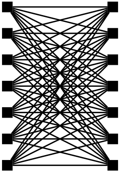
Parametrized Construction of Generalized Product Codes

product codes



staircase codes





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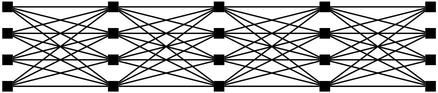

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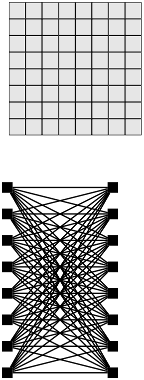

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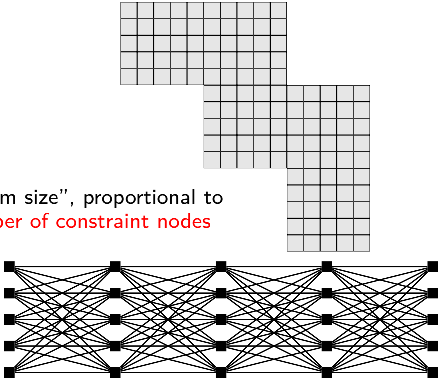

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Density Evolution

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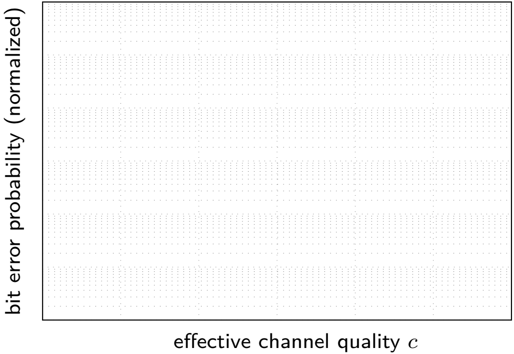

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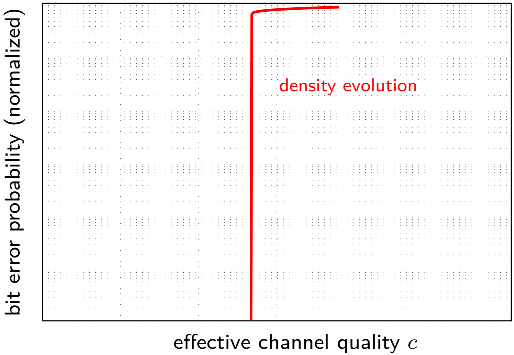

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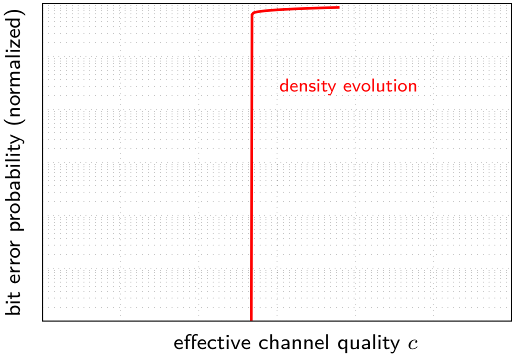

$$\boldsymbol{x}^{(\ell)} = \boldsymbol{\Psi}_{\geq t}(c\boldsymbol{B}\boldsymbol{x}^{(\ell-1)})$$

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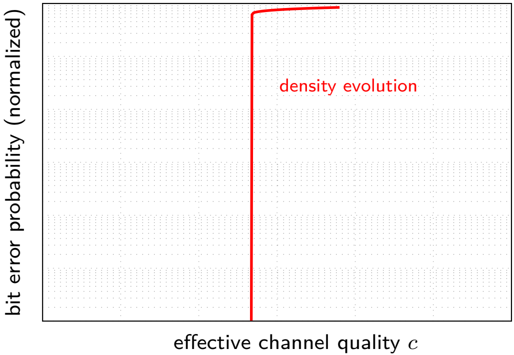

initial condition

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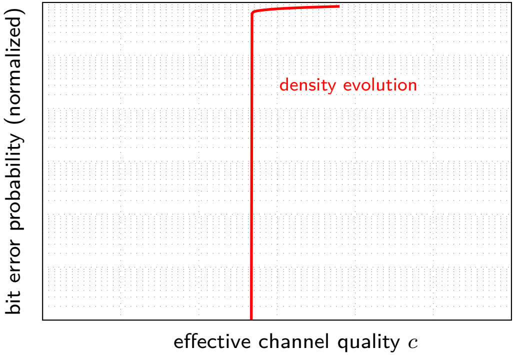

$\boldsymbol{B} \triangleq \gamma \boldsymbol{\eta}$

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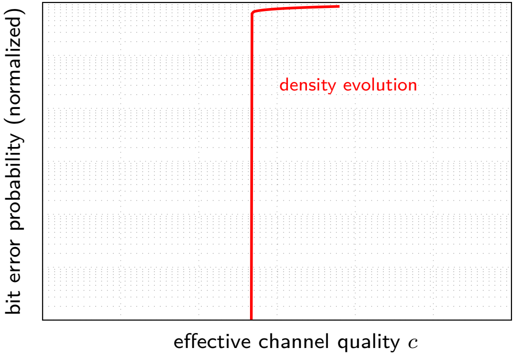
$$\boldsymbol{x}^{(\ell)} = \boldsymbol{\Psi}_{\geq t}(c\boldsymbol{B}\boldsymbol{x}^{(\ell-1)})$$

Density Evolution

- What happens asymptotically for $n \to \infty$?
- Let $p = c/n$ for $c > 0$, where c is the effective channel quality



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$\Psi_{\geq t}(x) \triangleq 1 - \sum_{i=0}^{t-1} \frac{x^i}{i!} e^{-x}$

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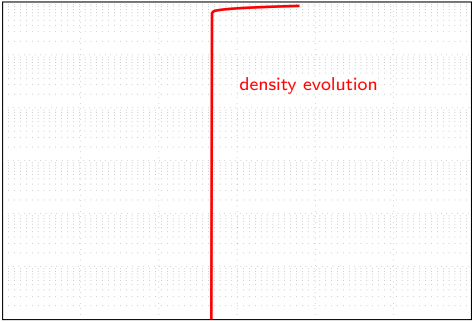
bit error probability (normalized)



effective channel quality c

$\boldsymbol{B} \triangleq \gamma \boldsymbol{\eta}$

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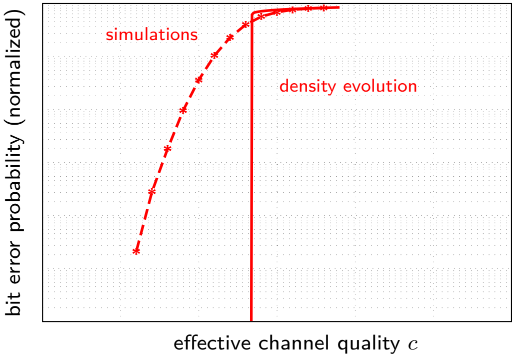

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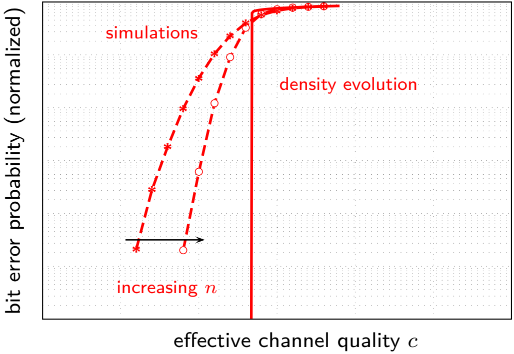

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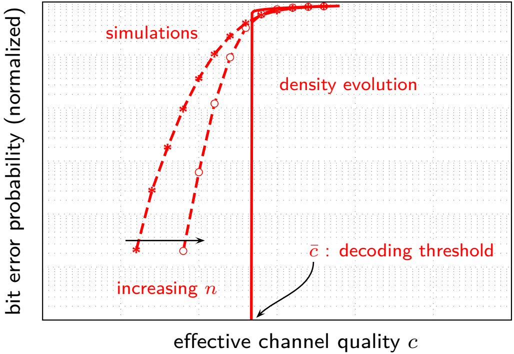

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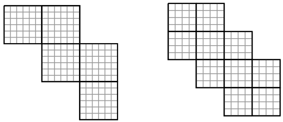
Comparison of Deterministic Codes and Ensembles

Comparison of Deterministic Codes and Ensembles

Deterministic



$$\boldsymbol{x}^{(\ell)} = \boldsymbol{\Psi}_{\geq t}(c\boldsymbol{B}\boldsymbol{x}^{(\ell-1)})$$

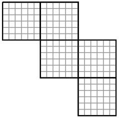
$$(\boldsymbol{B} = \gamma\boldsymbol{\eta})$$

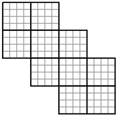
$$\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \frac{1}{3}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

staircase braided (simplified)

Comparison of Deterministic Codes and Ensembles

Deterministic





$$\boldsymbol{x}^{(\ell)} = \boldsymbol{\Psi}_{\geq t}(c\boldsymbol{B}\boldsymbol{x}^{(\ell-1)})$$

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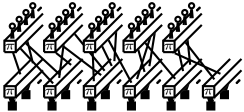
$$\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \frac{1}{3}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

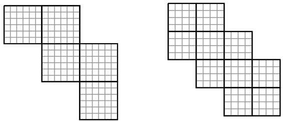
staircase braided (simplified)

Ensemble-Based [Jian et al., 2012]



Comparison of Deterministic Codes and Ensembles

Deterministic



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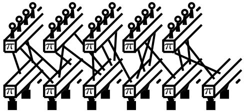
$(\boldsymbol{B} = \gamma\boldsymbol{\eta})$

$$\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \frac{1}{3}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

staircase — braided (simplified)

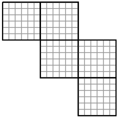
Ensemble-Based [Jian et al., 2012]

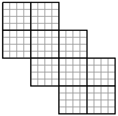


$$\boldsymbol{x}^{(\ell)} = \boldsymbol{\Psi}_{\geq t}(c\tilde{\boldsymbol{B}}\boldsymbol{x}^{(\ell-1)})$$

Comparison of Deterministic Codes and Ensembles

Deterministic





$$\boldsymbol{x}^{(\ell)} = \boldsymbol{\Psi}_{\geq t}(c\boldsymbol{B}\boldsymbol{x}^{(\ell-1)})$$

$$(\boldsymbol{B} = \gamma\boldsymbol{\eta})$$

$$\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \frac{1}{3}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

staircase braided (simplified)

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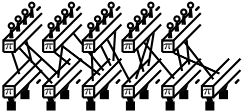


$$\boldsymbol{x}^{(\ell)} = \boldsymbol{\Psi}_{\geq t}(c\tilde{\boldsymbol{B}}\boldsymbol{x}^{(\ell-1)})$$

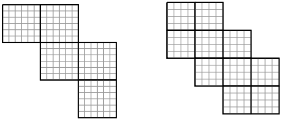
$$(\tilde{\boldsymbol{B}} = \boldsymbol{A}^\intercal\boldsymbol{A})$$

$$\boldsymbol{A} = \frac{1}{w}\begin{pmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

$\underbrace{\qquad\qquad}_{w}$ w (coupling width)

Comparison of Deterministic Codes and Ensembles

Deterministic



$$\boldsymbol{x}^{(\ell)} = \boldsymbol{\Psi}_{\geq t}(c\boldsymbol{B}\boldsymbol{x}^{(\ell-1)})$$

$$(\boldsymbol{B} = \gamma\boldsymbol{\eta})$$

$$\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \frac{1}{3}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

staircase braided (simplified)

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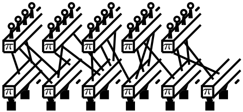


$$\boldsymbol{x}^{(\ell)} = \boldsymbol{\Psi}_{\geq t}(c\tilde{\boldsymbol{B}}\boldsymbol{x}^{(\ell-1)})$$

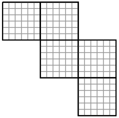
$$(\tilde{\boldsymbol{B}} = \boldsymbol{A}^{\intercal}\boldsymbol{A})$$

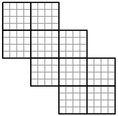
$$\frac{1}{4}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad \frac{1}{9}\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 2 & 1 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$w = 2$ $w = 3$

Comparison of Deterministic Codes and Ensembles

Deterministic





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staircase — braided (simplified)

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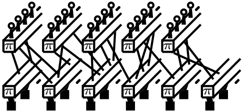

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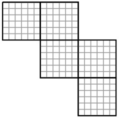
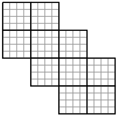
$w = 2$ — $w = 3$

- Density evolution equations have the same form

Comparison of Deterministic Codes and Ensembles

Deterministic





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staircase | braided (simplified)

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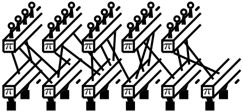


capacity-achieving at high rates over the binary symmetric channel

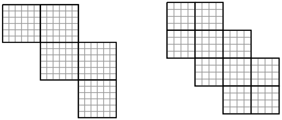
$$\frac{1}{4}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad \frac{1}{9}\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 2 & 1 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$w = 2$ | $w = 3$

- Density evolution equations have the same form

Comparison of Deterministic Codes and Ensembles

Deterministic



$$\boldsymbol{x}^{(\ell)} = \boldsymbol{\Psi}_{\geq t}(c\boldsymbol{B}\boldsymbol{x}^{(\ell-1)})$$

$$(\boldsymbol{B} = \gamma\boldsymbol{\eta})$$

$$\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \frac{1}{3}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

staircase — braided (simplified)

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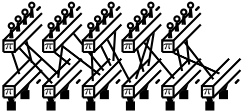

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$w = 2$ — $w = 3$

- Density evolution equations have the same form
- Different averaging due to matrices $\boldsymbol{B}$ and $\tilde{\boldsymbol{B}}$

Ensemble Performance via Deterministic Codes

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Can we "emulate" the ensemble density evolution with deterministic codes?

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Spatially-coupled ensemble performance for any fixed L and w

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$$\begin{matrix} 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ & 1 & 2 & 1 & & \\ & & 1 & 2 & 1 & \\ & & & 1 & 2 & 1 \\ & & & & 1 & 1 \end{matrix}$$

$\boldsymbol{P}$ for $L = 6$, $w = 2$

Ensemble Performance via Deterministic Codes

Can we "emulate" the ensemble density evolution with deterministic codes?

Spatially-coupled ensemble performance for any fixed L and w

1. Let $\boldsymbol{P} \triangleq w^2 \tilde{\boldsymbol{B}} = w^2 \boldsymbol{A}^\intercal \boldsymbol{A}$
2. Form $\boldsymbol{\eta}'$ from $\boldsymbol{P}$: replace entries $P_{i,j}$ by symmetric $w \times w$ matrices with $P_{i,j}$ ones per row/column

$$
\begin{array}{cccccc}
1 & 1 & & & & \\
1 & 2 & 1 & & & \\
 & 1 & 2 & 1 & & \\
 & & 1 & 2 & 1 & \\
 & & & 1 & 2 & 1 \\
 & & & & 1 & 1
\end{array}
$$

$\boldsymbol{P}$ for $L = 6$, $w = 2$

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$$
\begin{array}{cccccc}
\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} & 1 & & & & \\
1 & 2 & 1 & & & \\
 & 1 & 2 & 1 & & \\
 & & 1 & 2 & 1 & \\
 & & & 1 & 2 & 1 \\
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\end{array}
$$

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$$
\begin{matrix}
\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 1 & & & & \\
1 & \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} & 1 & & & \\
 & 1 & 2 & 1 & & \\
 & & 1 & 2 & 1 & \\
 & & & 1 & 2 & 1 \\
 & & & & 1 & 1
\end{matrix}
$$

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$$
\begin{array}{cccccccccccc}
1 & 0 & 1 & 0 & & & & & & & & \\
0 & 1 & 0 & 1 & & & & & & & & \\
1 & 0 & 1 & 1 & 1 & 0 & & & & & & \\
0 & 1 & 1 & 1 & 0 & 1 & & & & & & \\
 & & 1 & 0 & 1 & 1 & 1 & 0 & & & & \\
 & & 0 & 1 & 1 & 1 & 0 & 1 & & & & \\
 & & & & 1 & 0 & 1 & 1 & 1 & 0 & & \\
 & & & & 0 & 1 & 1 & 1 & 0 & 1 & & \\
 & & & & & & 1 & 0 & 1 & 1 & 1 & 0 \\
 & & & & & & 0 & 1 & 1 & 1 & 0 & 1 \\
 & & & & & & & & 1 & 0 & 1 & 0 \\
 & & & & & & & & 0 & 1 & 0 & 1
\end{array}
$$

$\boldsymbol{\eta}'$ (not unique)

Ensemble Performance via Deterministic Codes

Can we "emulate" the ensemble density evolution with deterministic codes?

Spatially-coupled ensemble performance for any fixed L and w

1. Let $\boldsymbol{P} \triangleq w^2 \tilde{\boldsymbol{B}} = w^2 \boldsymbol{A}^\intercal \boldsymbol{A}$
2. Form $\boldsymbol{\eta}'$ from $\boldsymbol{P}$: replace entries $P_{i,j}$ by symmetric $w \times w$ matrices with $P_{i,j}$ ones per row/column
3. $\eta_{2i,2j-1} = \eta'_{i,j}$ and $\eta_{2i-1,2j} = \eta'_{j,i}$ (create row/column array structure)

$$
\begin{array}{cccccccccccc}
1 & 0 & 1 & 0 & & & & & & & & \\
0 & 1 & 0 & 1 & & & & & & & & \\
1 & 0 & 1 & 1 & 1 & 0 & & & & & & \\
0 & 1 & 1 & 1 & 0 & 1 & & & & & & \\
 & & 1 & 0 & 1 & 1 & 1 & 0 & & & & \\
 & & 0 & 1 & 1 & 1 & 0 & 1 & & & & \\
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\end{array}
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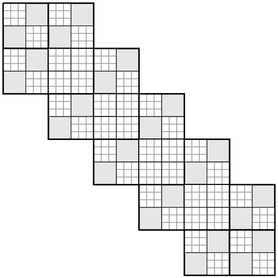

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Ensemble Performance via Deterministic Codes

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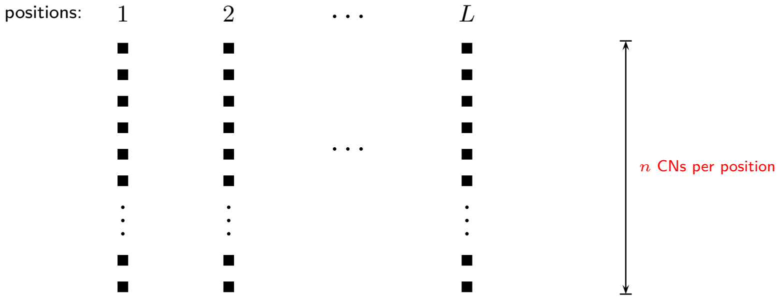

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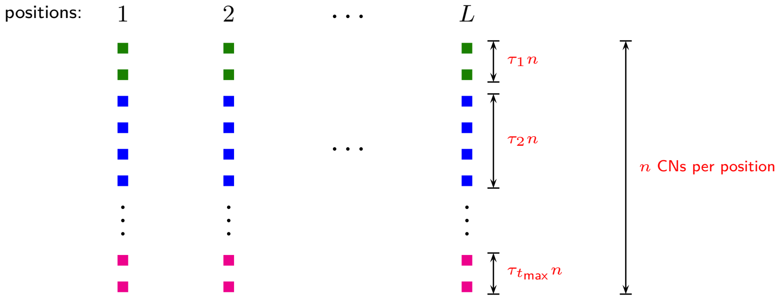

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$\Longrightarrow$ no asymptotic ($n \to \infty$, $\ell \to \infty$, large w) performance improvement possible

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Thank you!

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