Density Evolution and Error Floor Analysis for Staircase and Braided Codes

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Abstract

We analyze deterministically constructed (i.e., non-ensemble-based) codes in the waterfall and error floor region. The analysis directly applies to several forward error-correction (FEC) classes proposed for high-speed optical transport networks such as staircase [1] and braided codes [2, 4].

Motivation

- Product codes: each row and column of a rectangular array is a codeword in some component code (standardized in, e.g., ITU-T G.975).
- Recently, several classes of generalized product codes such as staircase and braided codes have been proposed (very appealing due to syndrome compression at high code rates $\implies$ low complexity [1]).
- We propose a construction that recovers these codes as special cases.
- Rigorous asymptotic density evolution analysis is possible which predicts the post-FEC bit error rate (BER) waterfall performance.
- We assume ℓ iterations of idealized hard-decision bounded-distance decoding over the binary symmetric channel with crossover probability p.
- Case study: comparison of staircase, braided, and half-braided codes.

Code Construction and Analysis

Parameters:

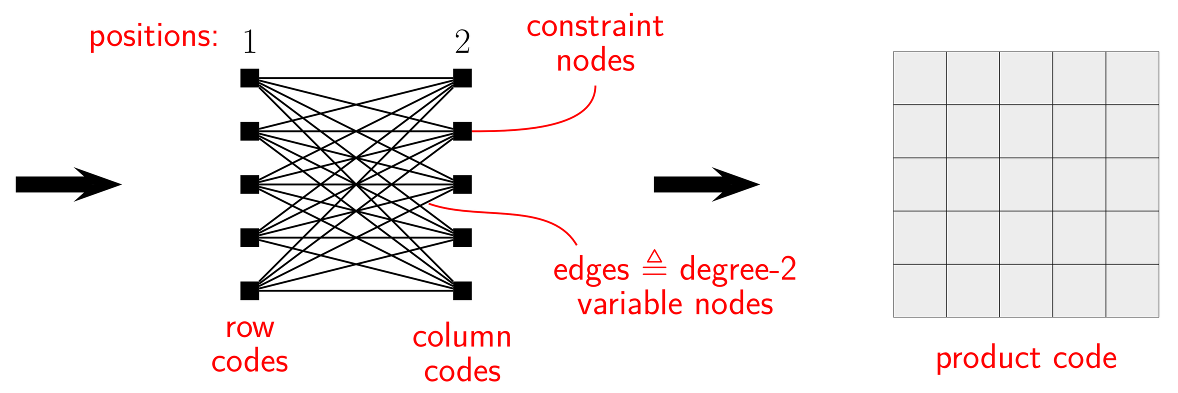
$\boldsymbol{\eta}$: binary symmetric $L \times L$ matrix, where L is the number of positions

n: number of constraint nodes (CNs) in the Tanner graph

Code construction: Place n/L CNs at each position. Connect each CNs at position i to each CN at position j (through a variable node) if $\eta_{i,j} = 1$.

Example:

- $\boldsymbol{\eta} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$
- $L = 2$
- $n = 10$



Density evolution [3]: Let $p = c/n$, $c > 0$, and $n \to \infty$. Then, BER $\approx p\mathbf{x}\boldsymbol{\eta}\mathbf{x}^\intercal / \|\boldsymbol{\eta}\|_F^2$ where $\|\boldsymbol{\eta}\|_F^2$ is the number of 1s in $\boldsymbol{\eta}$, $\mathbf{x} \triangleq (x_1^{(\ell)}, \dots, x_L^{(\ell)})$, and

$$x_i^{(\ell)} = \Psi_{\geq \mathsf{t}}\left(\frac{c}{L}\sum_{j=1}^{L} \eta_{i,j} x_j^{(\ell-1)}\right) \qquad \text{with } x_i^{(0)} = 1 \text{ for } i \in \{1, 2, \dots L\}.$$

Here, $\Psi_{\geq \mathsf{t}}(\lambda) = 1 - e^{-\lambda}\sum_{i=0}^{\mathsf{t}-1}\frac{\lambda^i}{i!}$ denotes the Poisson tail probability.

Error floor [1]: $\text{BER}_{\text{floor}} \approx s_{\min} M p^{s_{\min}} / B$ where parameters are defined below.

Staircase Codes | Braided Codes | Half-Braided Codes

We assume a BCH component code with length $n_c = 720$, dimension $k_c = 690$, and error-correcting capability $\mathsf{t} = 3$.

Structure of $\boldsymbol{\eta}$ for $L = 6$:

staircase

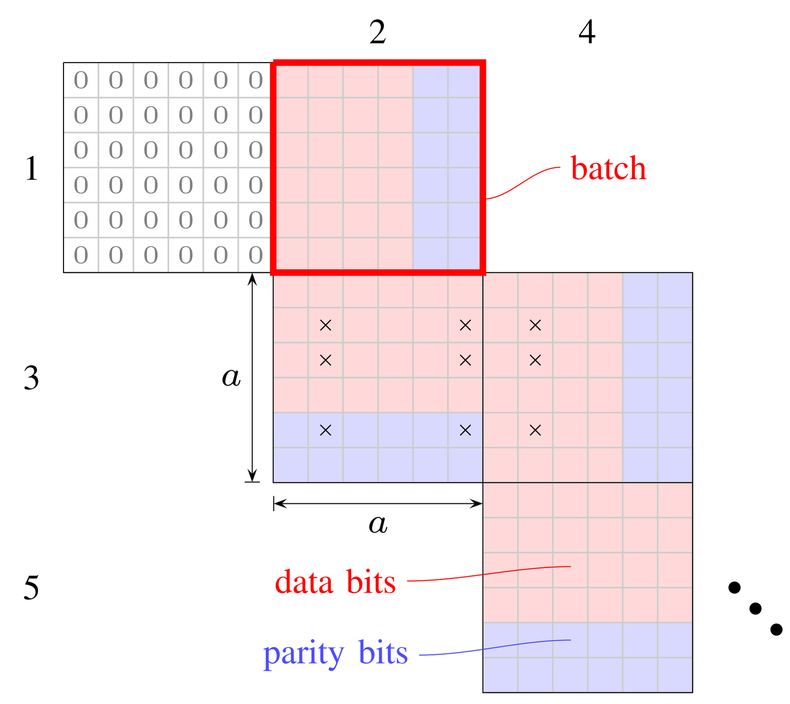
$$\begin{pmatrix} 0&1&0&0&0&0\\ 1&0&1&0&0&0\\ 0&1&0&1&0&0\\ 0&0&1&0&1&0\\ 0&0&0&1&0&1\\ 0&0&0&0&1&0 \end{pmatrix}$$

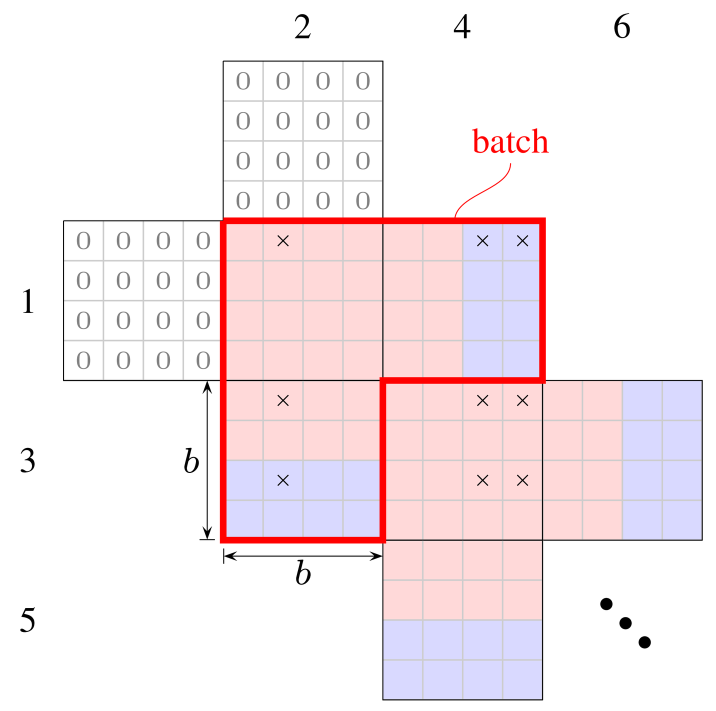
braided

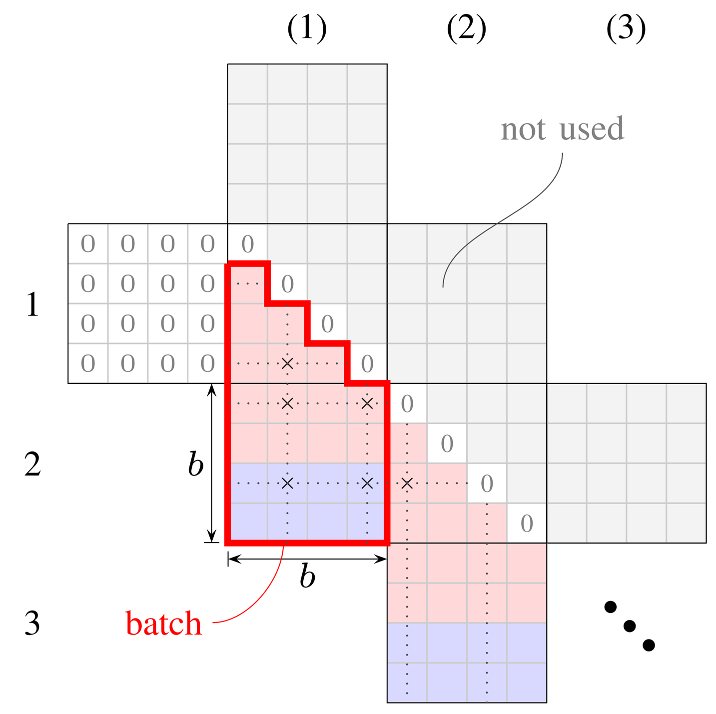
$$\begin{pmatrix} 0&1&0&1&0&0\\ 1&0&1&0&0&0\\ 0&1&0&1&0&1\\ 1&0&1&0&1&0\\ 0&0&0&1&0&1\\ 0&0&1&0&1&0 \end{pmatrix}$$

half-braided

$$\begin{pmatrix} 1&1&0&0&0&0\\ 1&1&1&0&0&0\\ 0&1&1&1&0&0\\ 0&0&1&1&1&0\\ 0&0&0&1&1&1\\ 0&0&0&0&1&1 \end{pmatrix}$$

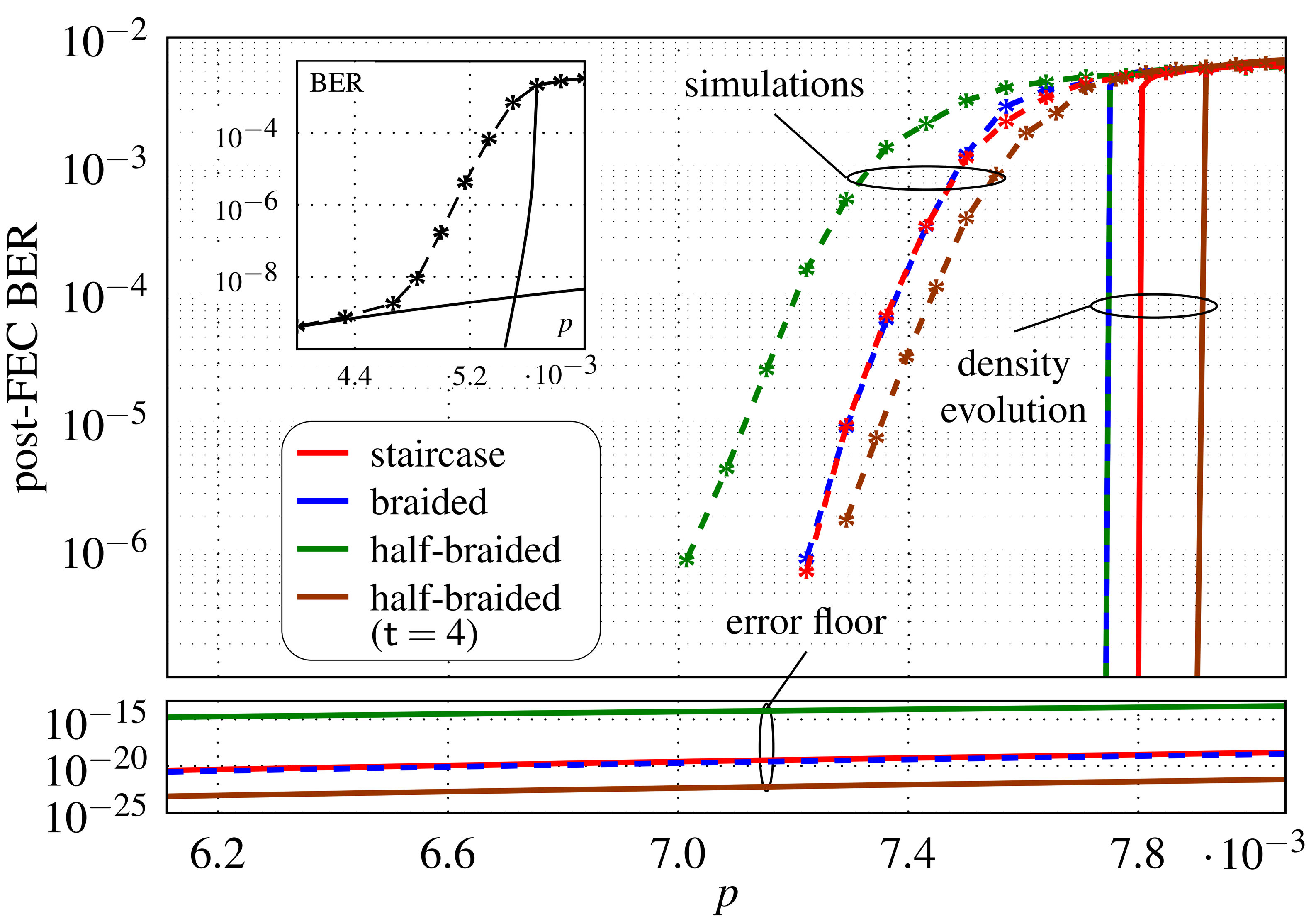






	Staircase Codes	Braided Codes	Half-Braided Codes
batch size B	$a^2 = n_c^2/4$	$3b^2 = n_c^2/3$	$(3b^2 - b)/2 = (n_c^2 - n_c)/6$
code rate R	$2k_c/n_c - 1 = 0.9167$	$2k_c/n_c - 1 = 0.9167$	$2(k_c - 1)/(n_c - 1) - 1 = 0.9166$
window decoder size W / iterations ℓ	8 / 8	6 / 8	6 / 16
decoding delay D	$1,036,800$	$1,036,800$	$517,680$
decoding schedule	row/column alternations	row/column alternations	all component codes at once
minimum stall pattern size $s_{\min}$	$(\mathsf{t}+1)^2 = 16$	$(\mathsf{t}+1)^2 = 16$	$(\mathsf{t}+1)(\mathsf{t}+2)/2 = 10$
stall pattern multiplicity M	$\binom{a}{\mathsf{t}+1}\left(\binom{2a}{\mathsf{t}+1} - \binom{a}{\mathsf{t}+1}\right)$	$\left(\binom{2b}{\mathsf{t}+1} - \binom{b}{\mathsf{t}+1}\right)^2 + 2\binom{b}{\mathsf{t}+1}\left(\binom{3b}{\mathsf{t}+1} - \binom{2b}{\mathsf{t}+1}\right)$	$\binom{2b}{\mathsf{t}+2} - \binom{b}{\mathsf{t}+2}$

Results



Inlet figure: $n_c = 600$, $k_c = 580$, $\mathsf{t} = 2$, $W = 5$, $\ell = 10$

Brown line: $n_c = 960$, $k_c = 920$, $\mathsf{t} = 4 \implies D = 920,640$, $s_{\min} = 15$

Conclusions

- Density evolution can be applied to several deterministic code classes that are relevant for fiber-optical communications.
- The analysis is useful for parameter tuning, optimization of window decoding schedules, or the design of new codes.
- Staircase and braided codes perform similarly, while half-braided codes can have better performance at a lower decoding delay.

References

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